

The Statistic Number of Pseudoinversions on the Colored Permutation Groups

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Abstract

We introduce the statistic number of i -pseudoinversions on the colored permutation groups. This statistic is more or less a generalization of the statistic number of inversions. The main motivation to study this statistic is that we can use it to define a number system and a numeral system on the colored permutation groups. By means of this statistic, we construct our number system, and a bijection between the set of positive integers and the colored permutation groups. We also deduce the generating function of the statistic number of pseudoinversions which is a generalization of the Poincaré polynomials of the symmetric group and of the hyperoctahedral group.

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1 Introduction

The colored permutation group $\mathcal{G}_{r,n}$ is the wreath product $Z_r \wr \mathcal{S}_n$ of the group Z_r of r^{th} roots of unity $1, e^{2\pi i \frac{1}{r}}, \dots, e^{2\pi i \frac{r-1}{r}}$ by the symmetric group \mathcal{S}_n . It is isomorphic to the imprimitive unitary reflection group $G(r, p, n)$. We represent an element π of $\mathcal{G}_{r,n}$ by

$$\pi = \begin{pmatrix} 1 & 2 & \dots & n \\ \xi_{k_1} \sigma(1) & \xi_{k_2} \sigma(2) & \dots & \xi_{k_n} \sigma(n) \end{pmatrix} \text{ with } \sigma \in \mathcal{S}_n \text{ and } \xi_{k_j} = e^{2\pi i \frac{k_j}{r}}.$$

A statistic over a group is a function from the group to the set of positive integers. One of the most studied statistic is the number of inversions on the symmetric group \mathcal{S}_n defined by $\text{inv } \sigma := \#\{(i, j) \in [n]^2 \mid i < j, \sigma(i) > \sigma(j)\}$. A well-known result on it is the equidistribution of inv and the statistic major index which was proved by Foata [3].

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The definition of this statistic is extended to the hyperoctahedral group $\mathcal{B}_n := Z_2 \wr \mathcal{S}_n$. Denoting $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ the set of the canonical vectors of \mathbb{R}^n , recall that \mathcal{B}_n acts on \mathbb{R}^n by

$$\pi \cdot \sum_{i=1}^n \lambda_i \mathbf{e}_i = \sum_{i=1}^n \lambda_i \xi_{k_i} \mathbf{e}_{\sigma(i)} \quad \text{with} \quad \lambda_1, \dots, \lambda_n \in \mathbb{R}.$$

The root system of \mathcal{B}_n is $\Phi_n := \{\pm \mathbf{e}_i, \pm \mathbf{e}_i \pm \mathbf{e}_j \mid 1 \leq i \neq j \leq n\}$, and its positive root system is $\Phi_n^+ := \{\mathbf{e}_k, \mathbf{e}_i + \mathbf{e}_j, \mathbf{e}_i - \mathbf{e}_j \mid k \in [n], 1 \leq i < j \leq n\}$. The statistic number of inversions on \mathcal{B}_n is then $\text{inv } \pi := \#\{v \in \Phi_n^+ \mid \pi \cdot v \in -\Phi_n^+\}$. After having defined the statistic major index on \mathcal{B}_n , Reiner established a refinement of the equidistribution of both statistics on this group [5, 6. A Bijection Relating $\text{maj}(\pi)$ and $\text{inv}(\pi)$].

The interest for investigating statistics on $\mathcal{G}_{r,n}$ recently arised. Bagno et al., for example, introduced the statistics (c, d) -descents and computed their distributions [1, Proposition 1.1.]. In this article, we introduce the statistic number of i -pseudoinversions defined by:

- If $\xi_{k_i} = 1$, then

$$\text{pinv}_i \pi = \#\{j \in \llbracket n - i + 1; n \rrbracket \mid \sigma(n - i + 1) > \sigma(j)\}.$$

- If $\xi_{k_i} \neq 1$, then

$$\begin{aligned} \text{pinv}_i \pi &= k_i + k_i \times \#\{j \in \llbracket n - i + 1; n \rrbracket \mid \sigma(n - i + 1) > \sigma(j)\} \\ &\quad + (k_i + 1) \times \#\{j \in \llbracket n - i + 1; n \rrbracket \mid \sigma(n - i + 1) < \sigma(j)\}. \end{aligned}$$

And the statistic number of pseudoinversions defined by $\text{pinv } \pi := \sum_{i=1}^n \text{pinv}_i \pi$.

Let us consider, for example, the element $\pi = \begin{pmatrix} 1 & 2 & 3 \\ 2 & \xi_1 1 & \xi_4 3 \end{pmatrix}$ of $\mathcal{G}_{5,3}$. We have $\text{pinv}_1 \pi = 4$, $\text{pinv}_2 \pi = 3$, $\text{pinv}_3 \pi = 1$, and $\text{pinv } \pi = 8$.

One can easily verify that for the cases $r = 1$ and $r = 2$, the statistic pinv is equal to the statistic inv of \mathcal{S}_n and \mathcal{B}_n respectively.

We can construct a number system by means of the cardinalities the groups $\mathcal{G}_{r,n}$ and the statistic pinv_i . Suppose that we take two sequences of strictly positive integers $\mathbf{Q} = (\mathbf{Q}_i)_{i \in \mathbb{N}}$ and $\mathbf{c} = (\mathbf{c}_i)_{i \in \mathbb{N}}$. Then, the pair (\mathbf{Q}, \mathbf{c}) is called a number system if every positive integer n has a single representation

$$n = \sum_{i=0}^k \alpha_i \mathbf{Q}_i \quad \text{with} \quad \alpha_i \in \llbracket 0; \mathbf{c}_i \rrbracket.$$

Cantor provided a condition for a pair of positive integer sequences to be a numeral system with tools of formal power series [2, §.2.]. Here we provide an usefull and lighter condition for us.

Proposition 1.1. *Let $\mathbf{Q} = (\mathbf{Q}_i)_{i \in \mathbb{N}}$ and $\mathbf{c} = (\mathbf{c}_i)_{i \in \mathbb{N}}$ be two sequences of strictly positive integers. The pair (\mathbf{Q}, \mathbf{c}) is a number system if and only if $\mathbf{Q}_0 = 1$ and*

$$\mathbf{Q}_k = \prod_{i=0}^{k-1} (1 + \mathbf{c}_i).$$

The proof of Proposition 1.1 is in Section 2. We deduce our number system.

Corollary 1.2. *Let $G_i = |\mathcal{G}_{r,i}| = r^i i!$ and $g_i = \max \text{pinv}_{i+1} = r(i+1) - 1$. Then $(G, g) = ((G_i)_{i \in \mathbb{N}}, (g_i)_{i \in \mathbb{N}})$ is a number system.*

Proof. We have

$$\prod_{i=0}^{k-1} (1 + g_i) = \prod_{i=0}^{k-1} r(i+1) = r^k k! = G_k.$$

Then we use Proposition 1.1. □

The factorial number system introduced by Laisant [4] corresponds exactly to the case $r = 1$ of our number system.

We can also construct a numeral system by means of the groups $\mathcal{G}_{r,n}$ and the statistic pinv_i . A numeral system is a mathematical notation for representing numbers of a given set. There exist several types of numeral systems depending on the historical context and geographical location. They play a central role in Coding Theory. Here we develop a numeral system on colored permutation groups. We use the notations $\alpha_k : \alpha_{k-1} : \dots : \alpha_1 : \alpha_0 := \sum_{i=0}^k \alpha_i G_i$ and

$$\langle G, g \rangle_k := \{ \alpha_k : \dots : \alpha_1 : \alpha_0 \mid \alpha_i \in \llbracket 0; g_i \rrbracket \}.$$

The numeral system stems from the following bijection.

Theorem 1.3. *The following application is bijective:*

$$g : \begin{array}{ccc} \mathcal{G}_{r,n} & \rightarrow & \langle G, g \rangle_{n-1} \\ \pi & \mapsto & \text{pinv}_n(\pi) : \text{pinv}_{n-1}(\pi) : \dots : \text{pinv}_1(\pi) \end{array}.$$

For example, $g \left(\begin{array}{ccc} 1 & 2 & 3 \\ 2 & \xi_1 1 & \xi_4 3 \end{array} \right) = 1 : 3 : 4$ which is equal to 69 in decimal system.

We prove Theorem 1.3 in Section 3. The Lehmer code is based on the case $r = 1$ of this numeral system. Still in the coding context, Vajnovszki provided also several permutation codes directly related to the Lehmer code [6].

We obtain the generating function of the statistic number of pseudoinversions. Define the q -analog of the number i by $[i]_q := 1 + q + \dots + q^{i-1}$.

Corollary 1.4. *The generating function of the statistic pinv on $\mathcal{G}_{r,n}$ is*

$$\sum_{\pi \in \mathcal{G}_{r,n}} q^{\text{pinv } \pi} = \prod_{i=1}^n [ri]_q.$$

Proof. The bijection of Proposition 1.3 implies that every element of $\mathcal{G}_{r,n}$ has a unique representation in $\langle G, g \rangle_{n-1}$. Then, we have

$$\sum_{\pi \in \mathcal{G}_{r,n}} q^{\text{pinv } \pi} = [\max \text{pinv}_n + 1]_q \dots [\max \text{pinv}_2 + 1]_q [\max \text{pinv}_1 + 1]_q = \prod_{i=1}^n [ri]_q.$$

□

Note that the Poincaré polynomials of the special cases \mathcal{S}_n and \mathcal{B}_n are $\prod_{i=1}^n [i]_q$ and $\prod_{i=1}^n [2i]_q$ respectively.

2 Proof of Proposition 1.1

In this section, we prove the condition for a pair of sequences of strictly positive integers (Q, c) to be a number system. We keep the notations $\alpha_k : \dots : \alpha_1 : \alpha_0 := \sum_{i=0}^k \alpha_i Q_i$ and $\langle Q, c \rangle_k := \{\alpha_k : \dots : \alpha_1 : \alpha_0 \mid \alpha_i \in \llbracket 0; c_i \rrbracket\}$.

Lemma 2.1. *Suppose that (Q, c) is a number system. Let $n = \alpha_k : \dots : \alpha_1 : \alpha_0$ be a positive integer. Then, $\alpha_k Q_k \leq n < (\alpha_k + 1)Q_k$.*

Proof. It is clear that $\alpha_k Q_k \leq n$. Suppose that $n \geq (\alpha_k + 1)Q_k$ which means $\sum_{i=0}^{k-1} \alpha_i Q_i \geq Q_k$.

Then, there exist λ_i such that $0 \leq \lambda_i \leq \alpha_i$ and $\lambda_{k-1} : \dots : \lambda_1 : \lambda_0 = 1 : \overbrace{0 : \dots : 0}^{k \text{ times}} : 0$. Which contradicts the unicity of the representation. \square

Lemma 2.2. *Consider two sequences of strictly positive integers $Q = (Q_i)_{i \in \mathbb{N}}$ and $c = (c_i)_{i \in \mathbb{N}}$. Then, the pair (Q, c) is a number system if and only if $Q_0 = 1$ and*

$$Q_k = \sum_{i=0}^{k-1} c_i Q_i + 1.$$

Proof. Suppose that (Q, c) is a number system. It is obvious that we must have $Q_0 = 1$. From the second inequality of Lemma 2.1, we deduce that

$$\sum_{i=0}^k c_i Q_i + 1 \leq (c_k + 1)Q_k \quad \text{i.e.} \quad \sum_{i=0}^{k-1} c_i Q_i + 1 \leq Q_k.$$

From the first inequality of Lemma 2.1, we deduce that the only possibility is the equality $\sum_{i=0}^{k-1} c_i Q_i + 1 = Q_k$.

Now, suppose that $Q_0 = 1$ and $Q_k = \sum_{i=0}^{k-1} c_i Q_i + 1$. Then one can uniquely construct every positive integer by induction:

$$\text{if } n = \sum_{i=0}^k \alpha_i Q_i, \text{ then } n + 1 = \sum_{i=0}^k \alpha_i Q_i + 1 \in \langle Q, c \rangle_{k+1}.$$

\square

We can now proceed to the proof of Proposition 1.1:

Proof. From Lemma 2.2, we deduce that (Q, c) is a number system if and only if $Q_0 = 1$ and

$$\begin{aligned} Q_k &= \sum_{i=0}^{k-1} c_i Q_i + 1 = c_{k-1} Q_{k-1} + \sum_{i=0}^{k-2} c_i Q_i + 1 = c_{k-1} Q_{k-1} + Q_{k-1} \\ &= (c_{k-1} + 1) Q_{k-1} = (c_{k-1} + 1)(c_{k-2} + 1) \dots (c_0 + 1) = \prod_{i=0}^{k-1} (1 + c_i). \end{aligned}$$

\square

3 Proof of Theorem 1.3

We begin with the following lemma.

Lemma 3.1. *We have $\text{pinv}_i(\mathcal{G}_{r,n}) = \llbracket 0; ri - 1 \rrbracket$ and*

$$\pi(i) \in \{\xi_k 1, \dots, \xi_k n\} \quad \text{if and only if} \quad \text{pinv}_{n-i+1}(\pi) \in \llbracket k(n-i+1); k(n-i+1) + n - i \rrbracket.$$

Proof. Use the definition of pinv_i . □

Let $s : \llbracket 1; n \rrbracket \times \llbracket 0; ri - 1 \rrbracket \rightarrow \llbracket 0; n - i \rrbracket$ be the function defined by

- $s(i, j) = j$ if $j \in \llbracket 0; n - i \rrbracket$,
- $s(i, j) = k(n - i + 1) + n - i - j$ if $j \in \llbracket k(n - i + 1); k(n - i + 1) + n - i \rrbracket$ (for $k \in \llbracket 1; r - 1 \rrbracket$).

Lemma 3.2. *We have $s(i, \text{pinv}_{n-i+1}(\pi)) = \#\{j \in \llbracket i + 1; n \rrbracket \mid \sigma(i) > \sigma(j)\}$.*

Proof. We use Lemma 3.1. Let $a = \#\{j \in \llbracket i + 1; n \rrbracket \mid \sigma(i) > \sigma(j)\}$:

- If $\xi_{k_i} = 1$, then $a = \text{pinv}_{n-i+1}(\pi) = s(i, \text{pinv}_{n-i+1}(\pi))$.
- If $\xi_{k_i} \neq 1$, then $\text{pinv}_{n-i+1}(\pi) = k_i + k_i a + (k_i + 1)(n - i - a)$ and

$$a = k_i(n - i + 1) + n - i - \text{pinv}_{n-i+1}(\pi) = s(i, \text{pinv}_{n-i+1}(\pi)).$$

□

Let \bar{g} be the function

$$\bar{g} : \begin{array}{ccc} \langle \mathbf{G}, \mathbf{g} \rangle_{n-1} & \rightarrow & \mathcal{G}_{r,n} \\ \alpha_{n-1} : \dots : \alpha_1 : \alpha_0 & \mapsto & \pi \end{array}$$

defined recursively as follows:

We have to calculate the $\pi = \begin{pmatrix} 1 & 2 & \dots & n \\ \xi_{k_1} \sigma(1) & \xi_{k_2} \sigma(2) & \dots & \xi_{k_n} \sigma(n) \end{pmatrix}$ corresponding to the number $\alpha_{n-1} : \dots : \alpha_1 : \alpha_0$. For that, we determine the order of the values of the $\sigma(i)$'s, and the values of k_i 's. The k_i 's are rather easy to calculate: For $k \in \llbracket 0; r - 1 \rrbracket$,

$$\text{if } \alpha_i \in \llbracket k(i + 1); k(i + 1) + i \rrbracket, \text{ then } k_{n-i} = k.$$

Let x_1, \dots, x_n be n variables. It remains to replace each variable in $x_1 > x_2 > \dots > x_n$ with the corresponding $\sigma(i)$. We implemente the following procedure from n down to 1:

We put $x_n \leftarrow \sigma(n)$ and obtain $x_1 > x_2 > \dots > \sigma(n)$.

For $\sigma(n - 1)$:

- If $s(n - 1, \alpha_1) = 1$, then $x_{n-1} \leftarrow \sigma(n - 1)$, and we obtain $x_1 > \dots > \sigma(n - 1) > \sigma(n)$.
- Else $x_{n-1} \leftarrow \sigma(n)$, $x_n \leftarrow \sigma(n - 1)$, and we obtain $x_1 > \dots > x_{n-2} > \sigma(n) > \sigma(n - 1)$.

Recursively, for $\sigma(j)$: Let $a = s(j, \alpha_{n-j})$.

- If $a = n - j$, then $x_j \leftarrow \sigma(j)$, and we obtain $x_1 > \dots > x_{j-1} > \sigma(j) > \overbrace{\dots}^{\text{followed by } n-j \text{ } \sigma(i)\text{'s}}$.
- Else, for all $i \in \llbracket n - j + 1; n - a \rrbracket$ we do $x_{i-1} \leftarrow x_i$, we put $x_{n-a} \leftarrow \sigma(j)$, and we obtain $x_1 > \dots > x_{j-1} > \overbrace{\dots}^{\text{followed by } n-j-a \text{ } \sigma(i)\text{'s}} > \sigma(j) > \overbrace{\dots}^{\text{followed by } a \text{ } \sigma(i)\text{'s}}$.

At the end of this process, we obtain a complete order of the $\sigma(i)$'s and get their values. Now, a strightforward calculation leads us to conclude that $g^{-1} = \bar{g}$.

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